

A SLICE THEOREM FOR QUIVERS WITH AN INVOLUTION

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ABSTRACT. We study the Luna slice theorem in the case of quivers with an involution or supermixed quivers as introduced by Zubkov in [6]. We construct an analogue to the notion of a local quiver setting described in [3]. We use this technique to determine dimension vectors of simple supermixed representations.

1. INTRODUCTION

Given a reductive algebraic group G and a G -representation V we can construct the algebraic quotient $V//G$, which is the affine variety corresponding to the ring of invariant polynomial functions $\mathbb{C}[V]^G$. The embedding $\mathbb{C}[V]^G \subset \mathbb{C}[V]$ gives rise to a quotient map $V \rightarrow V//G$.

The main problem in invariant theory is to describe the geometry of such a quotients. There are several questions that one can try to answer: What is the dimension of $V//G$? How do the fibers of $V//G$ look like? Is $V//G$ a smooth variety?

In complete generality a solution for these problems is unattainable but given restrictions on the groups or the representations one can expect some interesting partial results. In many cases one can find a certain class of couples (V, G) which share the same geometrical properties for their quotients. More precisely one can try to find classes that are closed under local behavior. By this we mean that if we have a couple (V, G) and a point $p \in V//G$ we can find another couple (V_p, G_p) of the same class such that there is an étale neighborhood of p that is locally isomorphic to an étale neighborhood of the zero point in $V_p//G_p$. Such a result simplifies the questions a lot because we can use this local result to reduce the questions about complicated representations to more simple representations.

For (G, V) a representation space of a quiver this was done by Procesi and Le Bruyn in [3]. Similar results have been obtained for representation spaces of preprojective algebras by Crawley-Boevey [1].

In this paper we will study the case of supermixed quivers. These were introduced and studied by Zubkov and Lopatin in [6],[7],[4] and are closely related to generalized quivers which were studied by Derksen and Weyman in [2]. First we will give a coordinate free description of a representation space of a supermixed quiver by means of involutions on semisimple algebras. Then we will extend the results by Derksen and Weyman to obtain a representation theoretic interpretation of the points in the representation spaces and in the quotient. This will enable us to formulate an extension of the result on local quivers by Procesi and Le Bruyn to supermixed quivers. To make full use of this result we will also determine which supermixed settings have simple representations.

In the rest of this paper all our varieties and algebra will be considered over \mathbb{C} but the results apply to every algebraically closed field of characteristic zero.

2. A QUICK REVIEW OF QUIVER SETTINGS

We briefly recall that a quiver Q consists of a set of vertices Q_0 , a set of arrows Q_1 and two maps $h, t : Q_1 \rightarrow Q_0$ which assign to each arrows its source and its tail. It is easy to see that that a quiver is uniquely defined by its Euler form.

A dimension vector of a quiver is a map $\alpha : Q_0 \rightarrow \mathbb{N}$. We will call a couple (Q, α) a quiver setting.

To every quiver setting we can associate a semisimple algebra, its standard left module

$$S_\alpha := \bigoplus_{v \in Q_0} \text{Mat}_{\alpha_v \times \alpha_v}(\mathbb{C}), \quad \mathbb{C}^\alpha := \bigoplus_{v \in Q_0} \mathbb{C}^{\alpha_v}$$

and an S_α -bimodule

$$\text{Rep}(Q, \alpha) = \bigoplus_{a \in Q_1} \text{Mat}_{\alpha_{h(a)} \times \alpha_{t(a)}}(\mathbb{C}).$$

Vice versa if S is a semisimple algebra and M an S -bimodule we can find a quiver setting (Q, α) unique up to isomorphism such that $(S, M) \cong (S_\alpha, \text{Rep}(Q, \alpha))$. The vertices in Q correspond to the maximal set of orthogonal idempotents $\{v_1, \dots, v_k\}$ in the center of S . The dimension vector is $\alpha_{v_i} = \sqrt{\dim v_i S}$ while the number of arrows from v_j to v_i is $\frac{\dim v_i M v_j}{\alpha_{v_i} \alpha_{v_j}}$.

We define the group GL_α as the group of invertible elements in S_α . This group has an action on $\text{Rep}(Q, \alpha)$ by conjugation: $m \mapsto gmg^{-1}$. This action has a categorical quotient:

$$\text{iss}(Q, \alpha) = \text{Rep}(Q, \alpha) // \text{GL}_\alpha.$$

In algebraic terms we construct the quotient as follows: let $\mathbb{C}[\text{Rep}(Q, \alpha)]$ be the ring of polynomial functions over $\text{Rep}(Q, \alpha)$. On this ring we have an action of GL_α coming from conjugation action on $\text{Rep}(Q, \alpha)$. The subring of functions that are invariant under this action is the ring of polynomial function over the categorical quotient:

$$\mathbb{C}[\text{iss}(Q, \alpha)] = \mathbb{C}[\text{Rep}(Q, \alpha)]^{\text{GL}_\alpha}.$$

A path of length k in a quiver is a sequence of arrows $p = a_1 \dots a_k$ with $t(a_i) = h(a_{i+1})$. We denote its head and tail as $h(p) = h(a_1)$ and $t(p) = t(a_k)$. A path the tail of which equals its head is called a cycle. A vertex is also called a path of length zero. A cycle is called primitive if the heads of all its arrows are different i.e. it runs through every vertex at most once. The path algebra $\mathbb{C}Q$ is the vector space with as basis all paths and its multiplication is the concatenation of paths if possible and zero if not. The path algebra is graded by the length of the paths and it is Morita equivalent to the tensor algebra $T_S M = S \oplus M \oplus M \otimes_S M \oplus \dots$.

The degree zero part of $\mathbb{C}Q$ can be embedded into S as the center: $\mathbb{C}Q_0 = Z(S)$. As such we can see \mathbb{C}^α as a left $\mathbb{C}Q_0$ -module. Any point $W \in M = \text{Rep}(Q, \alpha)$ can be identified with a left module of $\mathbb{C}Q$ such that the restriction to $\mathbb{C}Q_0$ is \mathbb{C}^α . Every arrow a will act as W_a between $\mathbb{C}^{\alpha_{t(a)}}$ and $\mathbb{C}^{\alpha_{h(a)}}$ and as zero between the rest. Therefore the points of $\text{Rep}(Q, \alpha)$ will be called representations of Q with dimension vector α .

On the other hand if W is a left $\mathbb{C}Q$ -module we can find bases for every space $vW, v \in Q_0$. If we express the action of arrows $a : t(a)W \rightarrow h(a)W$ as matrices according to these bases we get a representation of Q with dimension vector $\alpha_W : v \mapsto \dim vW$.

In this way we can speak of simple, semisimple and indecomposable representations of Q . Two points W and W' will give isomorphic representations if and only if they are in the same GL_α -orbit.

If W is a representation we can evaluate the path $p: W_p = W_{a_1} \dots W_{a_k}$. To every cycle c we can associate the map $f_c : \mathrm{Rep}(Q, \alpha) \rightarrow \mathbb{C} : W \mapsto \mathrm{Tr} W_c$. This map is invariant under the GL_α -action and in general every invariant map can be written in terms of these:

Theorem 2.1 (Le Bruyn-Procesi). The ring $\mathbb{C}[\mathrm{iss}(Q, \alpha)]$ is generated by functions of the form f_c where c is a cycle in Q .

The representation-theoretical interpretation of the quotient can be summarized as follows.

Theorem 2.2. The points in $\mathrm{iss}(Q, \alpha)$ are in one to one correspondence to isomorphism classes of the semisimple representations of Q in $\mathrm{Rep}(Q, \alpha)$. Two points in $\mathrm{Rep}(Q, \alpha)$ are mapped to the same point in $\mathrm{iss}(Q, \alpha)$ if and only if they have the same semisimplification.

3. DUALIZING STRUCTURES AND SUPERMIXED SETTINGS

Let S be a finite dimensional semisimple algebra and let M be an S -bimodule.

Definition 3.1. A *dualizing structure* on (S, M) consists of two linear involutions $*$: $S \rightarrow S$ and $*$: $M \rightarrow M$ and a bilinear form $\langle \cdot, \cdot \rangle : \mathbb{C}^\alpha \times \mathbb{C}^\alpha \rightarrow \mathbb{C}$, satisfying the following compatibility relations:

- $\forall a, b \in S_\alpha : (ab)^* = b^*a^*$.
- $\forall a, b \in S_\alpha : \forall m \in M : (amb)^* = b^*m^*a^*$.

The *dualizing group* of S is the group of elements for which the inverse and the involution coincide:

$$D(S) = \{g \in S : gg^* = 1\}$$

while the *dualizing subspace* of M is the subspace

$$D(M) = \{v \in M : v^* = v\}$$

The group $D(S)$ has an action on $D(M)$ by conjugation: $v \mapsto gvg^*$ because $(gvg^*)^* = gvg^*$.

We can turn this data into the language of quivers. First we are going to impose the dualizing structure on a quiver setting. An involution $*$ on S_α restricts to an involution of the center which is a ring-automorphism. This maps idempotents to idempotents so we get an involution ϕ on the set of vertices $\{v_i\}$.

Given ϕ we can construct a standard involution on S_α :

$$s \mapsto s^\dagger \text{ with } (s^\dagger)_v = s_\phi(v)^\top$$

where the transpose is taken according to the identification $S_\alpha = \oplus_{v \in Q_0} \text{Mat}_{\alpha_v \times \alpha_v}(\mathbb{C})$. The composition of $*$ and \dagger gives us an automorphism of S_α which is internal because all idempotents are fixed. Therefore we can say that there exists a $g \in S_\alpha$ such that

$$(s^*)_v = g_v s_{\phi(v)}^\top g_v^{-1}$$

The fact that $*^2 = \text{id}$ implies that

$$((s^*)^*)_v = g_v g_{\phi(v)}^{-\top} s_v g_{\phi(v)}^\top g_{\phi(v)}^{-1} = s_v$$

so $g_v = \epsilon_v g_{\phi(v)}^\top$ for some scalars ϵ_v and $\epsilon_v \epsilon_{\phi(v)} = 1$. Because we only use the g_v for conjugation they are determined up to a scalar themselves. This allows us to chose $\epsilon_v = \pm 1$ and hence we also have that $\epsilon_v = \epsilon_{\phi(v)}$.

Now we will tackle the dualizing structures on $M = \text{Rep}(Q, \alpha)$. We suppose that the involution on S_α is in the form above and that ϕ is the corresponding involution on the vertices Q_0 . Choose a second involution $\phi : Q_1 \rightarrow Q_1$ such that $h(\phi(a)) = \phi(t(a))$ and if $h(a) = \phi(t(a))$ we demand that $\phi(a) = a$. Given this we can construct a 'standard' dualizing structure on $\text{Rep}(Q, \alpha)$.

$$m \mapsto m^\dagger \text{ with } (m^\dagger)_a = \epsilon_{t(a)} g_{h(a)} m_{\phi(a)}^\top g_{t(a)}^{-1}.$$

As in the case of S_α , we can compose the involution $*$: $\text{Rep}(Q, \alpha) \rightarrow \text{Rep}(Q, \alpha)$ we want to study with this standard involution to obtain an automorphism $*^\dagger$ of $\text{Rep}(Q, \alpha)$ as an S_α -module. By Schur's lemma there are coefficients σ_{ab} , $a, b \in Q_1$ such that $(m^{*\dagger})_a = \sum_{b \in Q_1} \sigma_{ab} m_b$. Also note that σ_{ab} is only nonzero if $h(a) = h(b)$ and $t(a) = t(b)$.

By taking linear combinations of the arrows, we can diagonalize σ and this gives the following formula for $*$:

$$(m^*)_a = \sigma_a \epsilon_{t(a)} g_{h(a)} m_{\phi(a)}^\top g_{t(a)}^{-1}$$

The fact that $*$ is an involution implies that $\sigma_a \sigma_{\phi(a)} = 1$ and rescaling ensures us that we can suppose $\sigma_a = \pm 1$.

All this rewriting can be summarized in the terminology of supermixed quivers.

Definition 3.2. A supermixed quiver \mathbb{Q} consists of a quiver Q , two involutions ϕ (one on the vertices and one on the arrows) and two sign maps $\epsilon : Q_0 \rightarrow \{\pm 1\}$ and $\sigma : Q_1 \rightarrow \{\pm 1\}$. which satisfy:

- M1 $h(\phi(a)) = \phi(t(a))$,
- M2 $h(a) = h(\phi(a)) \Rightarrow a = \phi(a)$,
- M3 $\epsilon_v \epsilon_{\phi(v)} = \sigma_a \sigma_{\phi(a)} = 1$.

We say that (\mathbb{Q}, γ) is a supermixed setting if $\gamma : Q_0 \rightarrow \sqcup_{n \in \mathbb{N}} \text{GL}_n(\mathbb{C})$ is a map that assigns to each vertex a matrix such that $g_v = \epsilon_v g_{\phi(v)}^\top$. To γ we can naturally associate a dimension vector which assigns to every vertex v the dimension of γ_v . We will denote this dimension vector by $\bar{\gamma}$, but we will leave the bar in cases where the dimension vector is just used as an index f.i. we will write S_γ instead of $S_{\bar{\gamma}}$.

Every supermixed quiver setting defines involutions $*$ on the algebra S_γ and on the space $\text{Rep}(\mathbb{Q}, \gamma)$:

$$\begin{aligned} (s^*)_v &= \gamma_{\phi(v)} s_{\phi(v)}^\top \gamma_{\phi(v)} \\ (m^*)_a &= \sigma_a \epsilon_{h(a)} \gamma_{t(a)}^\top m_{\phi(a)}^\top \gamma_{h(a)} \end{aligned}$$

The couple $(S_\gamma, \text{Rep}(\mathbb{Q}, \gamma))$ with the involutions above is called the dualizing structure coming from the supermixed quiver setting.

A vertex with $v = \phi(v)$ is called *orthogonal* if $\epsilon_v = 1$ and *symplectic* if $\epsilon_v = -1$, a vertex for which $v \neq \phi(v)$ is called *general*.

An arrow with $a = \phi(a)$ is called *symmetric* if $\sigma_a = 1$ and *antisymmetric* if $\sigma_a = -1$, an arrow for which $a \neq \phi(a)$ is called *general*.

The terminology introduced above is an adaptation from the one introduced by Zubkov in [6]. Instead of partitioning the vertices into sets of size one and two and looking at the double quiver we use involutions. Also we use the γ for the identification between the vector space and their duals instead of the more explicit approach in [6]. The translation between the two notations is straightforward, but our adaptation makes it easier to describe the results we obtained concerning local quiver settings.

Theorem 3.1. Every couple (S, M) with a dualizing structure is isomorphic to a dualizing structure coming from a normalized supermixed quiver setting.

Proof. This follows from the discussion above. \square

There is a redundancy in the definition of the supermixed settings.

Theorem 3.2. Two supermixed settings (\mathbb{Q}_1, γ_1) and (\mathbb{Q}_2, γ_2) have isomorphic dualizing structures if the underlying dimension vectors are the same and the sign maps have identical values on the fixed vertices and arrows.

$$\begin{aligned}\bar{\gamma}_1 &= \bar{\gamma}_2 \\ v = \phi(v) &\Rightarrow \epsilon_1(v) = \epsilon_2(v) \\ a = \phi(a) &\Rightarrow \sigma_1(a) = \sigma_2(a)\end{aligned}$$

Proof. If we apply a base change $h \in S_\alpha$ then γ_v will transform as

$$(\gamma')_v = h_v \gamma_v h_{\phi(v)}^\top.$$

This means that if $v = \phi(v)$ we can transform γ_v to the identity matrix if $\epsilon_v = 1$ and to the standard symplectic matrix $\Lambda_{2n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\oplus n}$ if $\epsilon_v = -1$. If $v \neq \phi(v)$ we can change γ_v to id (and hence $\gamma_{\phi(v)}$ to $\epsilon_v \text{id}$). So we see that the isomorphism class of the involution does not depend on the exact nature of γ only on $\bar{\gamma}$ and ϵ .

We can also multiply the γ_v with scalars

$$(\gamma')_v = \lambda_v \gamma_v$$

without changing the involution. This changes ϵ_v into $\epsilon'_v = \epsilon_v \lambda_v / \lambda_{\phi(v)}$, so only the values of ϵ on the fixed vertices are important. Keep in mind that σ_a also changes to $\sigma_a \lambda_{h(\phi(a))} / \lambda_{h(a)}$.

Finally we can alter the sign of σ_a if $a \neq \phi(a)$ by a base change on the vertices. \square

We can change ϵ, σ and γ without changing the isomorphism class of $(S_\gamma, \text{Rep}(\mathbb{Q}, \gamma))$ such that

- $v \neq \phi(v) \Rightarrow \epsilon_v = 1$,
- $a \neq \phi(a) \Rightarrow \sigma_a = 1$,

- $\gamma_v = \text{id}_{\bar{\gamma}_v}$ if $\epsilon_v = 1$ and $\gamma_v = \Lambda_{\bar{\gamma}_v}$ if $\epsilon_v = -1$.

If these conditions are met we speak of a *strict* supermixed setting, one can prove that there is a one to one correspondence between strict supermixed settings and isomorphism classes of dualizing structures. In the next sections, however we will need a more flexible description of dualizing structures so we will also allow non-strict settings.

The dualizing group of a supermixed quiver setting (\mathbb{Q}, γ) is by definition the group

$$\{h \in S_\gamma \mid \forall v \in Q_0 : h_v \gamma_v h_{\phi(v)}^\top \gamma_v^{-1} = 1\}$$

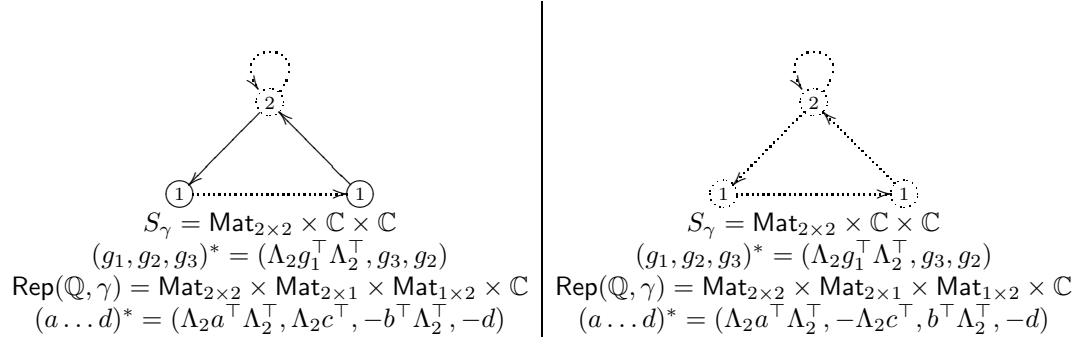
because up to isomorphism we can bring the supermixed setting in its strict form and we can conclude that

$$\text{DGL}_\gamma := D(S_\gamma) \cong \prod_{\{v, \phi(v)\} \in Q_0} \begin{cases} O_{\gamma_v} & v = \phi(v) \text{ and } \epsilon_v = 1 \\ Sp_{\gamma_v} & v = \phi(v) \text{ and } \epsilon_v = -1 \\ GL_{\gamma_v} & v \neq \phi(v). \end{cases}$$

This explains the terminology for the vertices.

We will depict supermixed quiver settings in the following way. We will consider the $*$ operation as a reflection around the vertical axis. In this way it is obvious to identify the duals of the vertices and arrows. The arrows with $\sigma_a = -1$ and the vertices with $\epsilon_v = -1$ will be drawn with dashed lines. If needed we will put the expression of γ_v inside the vertex but if we don't do this we suppose that $\gamma_v = \text{id}$ if $v = \phi_v$ and $\epsilon_v = 1$ or $v \neq \phi$ and v is the leftmost vertex of the pair (v, ϕ_v) . If $v = \phi(v)$ and $\epsilon_v = -1$ we suppose that $\Lambda_{\bar{\gamma}_v}$.

The two supermixed quiver settings below describe isomorphic dualizing structures, but only the left setting is strict.



4. ϵ -MIXED MODULES

Given a quiver setting (Q, α) one can interpret the spaces $\text{Rep}(Q, \alpha)$ and $\text{iss}(Q, \alpha)$ as classifying the $\mathbb{C}Q$ -module structures that one can put on \mathbb{C}^α . We will now try to do a similar thing for supermixed quiver settings. The discussion below matches the discussion of symmetric quivers in [2].

Given a a supermixed quiver setting (\mathbb{Q}, γ) we define a bilinear form on the space $\mathbb{C}^{\bar{\gamma}} := \bigoplus_v \mathbb{C}^{\bar{\gamma}_v}$:

$$\langle x, y \rangle := \sum_v x_v \gamma_v y_{\phi(v)}^\top.$$

If we identify ϵ with the element $\sum_v \epsilon_v v \in \mathbb{C}Q$ we see that this form has a special property: it is ϵ -commuting

$$\forall x, y \in \mathbb{C}^\gamma : \langle x, y \rangle = \langle y, \epsilon x \rangle.$$

This bilinear form can be used to define an adjoint: if $\varphi : \mathbb{C}^\gamma \rightarrow \mathbb{C}^\gamma$ we set

$$\forall x, y \in \mathbb{C}^\alpha : \langle \varphi x, y \rangle = \langle x, \varphi^\sharp y \rangle$$

Every $W \in \text{Rep}(\mathbb{Q}, \gamma)$ gives us a left $\mathbb{C}Q$ -module structure $\rho_W : \mathbb{C}Q \rightarrow \text{End}(\mathbb{C}^{\bar{\gamma}})$ on $\mathbb{C}^{\bar{\gamma}}$. The adjoint allows us to express the action on the dual module: ρ_W^\sharp . This is a right module so we cannot see this dual module as a point in $\text{Rep}(\mathbb{Q}, \gamma)$.

To do this we put an anti-automorphism $*$ on the path algebra

$$* : \mathbb{C}Q \rightarrow \mathbb{C}Q : \begin{cases} v \rightarrow \phi(v) \\ a \rightarrow \sigma_a \epsilon_{h(a)} \phi(a) \end{cases} \quad \text{and } \forall x, y \in \mathbb{C}Q : (xy)^* = y^* x^*.$$

This anti-automorphism is not an involution but it satisfies

$$\forall x \in \mathbb{C}Q : x^{**} = \epsilon x \epsilon$$

This anti-automorphism allows us to turn left $\mathbb{C}Q$ -modules into right $\mathbb{C}Q$ -module: $\rho_*(u) := \rho(u^*)$. Therefore, given the left module $W \in \text{Rep}(\mathbb{Q}, \gamma)$, we can consider the left module $W^* \in \text{Rep}(\mathbb{Q}, \gamma)$ corresponding to $(\rho_W^\sharp)^*$. This gives us an involution on $\text{Rep}(\mathbb{Q}, \gamma)$ and it is easy to check that this is in fact the same as the involution coming from the original dualizing structure.

All this warrants the following definition: an ϵ -mixed module is a left $\mathbb{C}Q$ -module V together with a nondegenerate bilinear map $\langle, \rangle : V \times V \rightarrow \mathbb{C}$ satisfying

- $\forall v, w \in V : \langle v, w \rangle = \langle w, \epsilon v \rangle$ (ϵ -commuting),
- $\forall v, w \in V : \forall x \in \mathbb{C}Q : \langle xv, w \rangle = \langle v, x^* w \rangle$ ($*$ -compatible).

If the $*$ -compatibility only holds for $\mathbb{C}Q_0$ instead of the whole $\mathbb{C}Q$ we will speak of *almost ϵ -mixed modules*.

For every (almost) ϵ -mixed $\mathbb{C}Q$ -module we can choose bases (b_i^v) in every vW and use these to construct a supermixed quiver setting (\mathbb{Q}, γ) with

$$(\gamma_v)_{ij} = \langle b_i^v, b_j^v \rangle$$

The ϵ -commutativity of \langle, \rangle ensures that whatever bases we take we will end up with a supermixed quiver setting that gives the same dualizing structure.

According to these bases we can express every a as a matrix W_a . If W is ϵ -mixed then (W_a) is in fact an element of $\text{DRep}(\mathbb{Q}, \gamma)$. Vice versa if $W \in \text{DRep}(\mathbb{Q}, \gamma)$ we can build an ϵ -mixed module out of it in the usual way.

Furthermore two (almost) ϵ -mixed modules are called ϵ -isomorphic if there is a module morphism between them that preserves the bilinear form. The group of these isomorphisms is DGL_γ and it is easy to check that $V, W \in \text{DRep}(\mathbb{Q}, \gamma)$ correspond to ϵ -isomorphic modules if and only if they are in the same orbit under DGL_γ . We can conclude:

Theorem 4.1. The DGL_γ -orbits in $\text{DRep}(\mathbb{Q}, \gamma)$ classify the ϵ -mixed modules structures on \mathbb{C}^γ , \langle, \rangle up to ϵ -isomorphism.

A second theorem is less trivial, but its proof can easily be adapted from [2][Thm 2.6].

Theorem 4.2. Two ϵ -mixed modules are isomorphic as ϵ -mixed modules if and only if they are isomorphic as $\mathbb{C}Q$ -modules.

$$\forall m \in \text{DRep}(\mathbb{Q}, \gamma) : \text{DGL}_\gamma m = \text{GL}_\alpha m \cap \text{DRep}(\mathbb{Q}, \gamma)$$

The ϵ -mixed structure is not only compatible with isomorphisms but it is also compatible with degenerations:

Theorem 4.3. If V is an ϵ -mixed module then its semisimplification as a $\mathbb{C}Q$ -module is also an ϵ -mixed module.

Proof. We prove this by induction on the length of the composition series of V . If V is simple then the statement is trivially true. Now suppose $S \subset V$ is a simple submodule then we also have a projection between the dual modules $V = V^* \rightarrow S^*$.

The kernel of this projection is $S^\perp = \{v \in V | \forall w \in S : \langle v, w \rangle = 0\}$. There are two possibilities: $S \cap S^\perp = 0$ or $S \subset S^\perp$. In the first case \langle, \rangle_S and $\langle, \rangle_{S^\perp}$ are nondegenerate and $V = S \oplus S^\perp$. By the induction hypothesis $S^{\perp ss}$ admits an ϵ -mixed structure and hence $V^{ss} = S \oplus S^{\perp ss}$ as well.

In the second case $\langle, \rangle_{S^\perp}$ is degenerate but it becomes nondegenerate if we quotient out $(S^\perp)^\perp = S$. By the induction hypothesis $(S^\perp/S)^{ss}$ admits an ϵ -mixed structure we can use this structure to put an ϵ -mixed structure on $V^{ss} = S \oplus (S^\perp/S)^{ss} \oplus S^*$:

$$\langle (s_1, t_1, u_1), (s_2, t_2, u_2) \rangle = u_2(s_1) + u_1(s_2) + \langle t_1, t_2 \rangle_{(S^\perp/S)^{ss}}$$

It is easy to see that the action of $\mathbb{C}Q$ is compatible with this form. \square

Apart from the orbits themselves we are also interested in the quotient space of the orbits.

$$\text{Diss}(\mathbb{Q}, \gamma) = \text{DRep}(\mathbb{Q}, \gamma) // \text{DGL}_\gamma$$

The main result we can use for this was proven by Zubkov in [6] and is stated in terms of invariants

Theorem 4.4. The DGL_γ -invariant functions on $\text{DRep}(\mathbb{Q}, \gamma)$ come from GL_γ -invariant function on $\text{Rep}(Q, \gamma)$: the diagram below is commutative

$$\begin{array}{ccc} \mathbb{C}[\text{Rep}(Q, \gamma)] & \longrightarrow & \mathbb{C}[\text{DRep}(\mathbb{Q}, \gamma)] \\ \uparrow & & \uparrow \\ \mathbb{C}[\text{Rep}(Q, \gamma)]^{\text{GL}_\gamma} & \longrightarrow & \mathbb{C}[\text{DRep}(\mathbb{Q}, \gamma)]^{\text{DGL}_\gamma} \end{array}$$

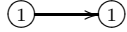
The relation between the quotient space and the representation theory is stated in this corollary

corollary 4.1. The points in $\text{Diss}(\mathbb{Q}, \gamma)$ parameterize the semisimple ϵ -mixed module up to isomorphism.

Proof. Every fiber of the map $\text{DRep}(\mathbb{Q}, \gamma) \rightarrow \text{Diss}(\mathbb{Q}, \gamma)$ contains a semisimple representation: suppose W is in the fiber of p and W^{ss} is the semisimplification of W . Because of theorem 4.3 W^{ss} can be considered as sitting inside $\text{DRep}(\mathbb{Q}, \gamma)$. As it is the semisimplification of W it has the same values as W on the GL_γ -invariant functions and hence also on the DGL_γ -invariant functions. As a consequence W and W^{ss} are in the same fiber. Also two non-isomorphic semisimples cannot be in the same fiber because their orbits are closed (they are intersections of closed GL_γ -orbits with $\text{DRep}(\mathbb{Q}, \gamma)$). \square

An ϵ -submodule is a submodule $W \subset V$ such that $\langle, \rangle|_W$ is non degenerate, an ϵ -mixed module is ϵ -irreducible if it has no nontrivial supermixed submodules. It is clear that it is possible to define the direct sum of two ϵ -mixed modules. If $W \subset V$ is an ϵ -submodule, we can define the submodule $W^\perp := \{v \in V | \forall w \in W : \langle v, w \rangle = 0\}$. The nondegeneracy of W makes that $W \cap W^\perp = 0$ and the compatibility of \langle, \rangle with the involution makes that W^\perp is also an ϵ -mixed representation, so $V = W \oplus W^\perp$.

It is not true that an ϵ -irreducible module is always semisimple as a $\mathbb{C}Q$ -module, so its orbit might not be closed in $\mathbf{DRep}(\mathbb{Q}, \gamma)$. An example of this phenomenon is the supermixed quiver setting:



All ϵ -mixed representations of this setting are ϵ -irreducible but only the one that assigns zero to the arrow is semisimple. Hence it is more interesting to restrict our attention to ϵ -irreducible that are also semisimple. Such modules will be called ϵ -simple and every semisimple ϵ -mixed module is a direct sum of ϵ -simple modules.

Theorem 4.5. Suppose that V is ϵ -simple. There are 3 possibilities:

- (1) V is a simple $\mathbb{C}Q$ -module,
- (2) $V \cong W \oplus W^*$ where W is a simple $\mathbb{C}Q$ -module and W and W^* are isomorphic modules.
- (3) $V \cong W \oplus W^*$ where W is a simple $\mathbb{C}Q$ -module and W and W^* are non-isomorphic modules.

Proof. If we are not in the first case, there exists a proper simple submodule $W \subset V$. As $W^\perp \cap W$ is a submodule of W , $W^\perp \cap W = W$ and W is perpendicular to itself (it cannot be that $W^\perp \cap W = 0$ because then W would be an ϵ -simple submodule).

Because V is semisimple we know that $V \cong W \oplus W^\perp/W \oplus W^*$. But the bilinear form on W^\perp/W is nondegenerate so W^\perp/W is an ϵ -submodule of V and hence 0. This proves that there are 3 possible cases. \square

We will call these types of ϵ -simple modules (1) orthogonal, (2) symplectic and (3) general, because the ϵ -automorphism groups of such modules are (1) O_1 , (2) \mathbf{Sp}_2 , (3) \mathbf{GL}_1 .

The theorem implies that we can decompose a semisimple ϵ -mixed representation as follows

$$\bigoplus_{1 \leq \ell \leq k_1} S_\ell^{e_\ell} \quad \bigoplus_{k_1 < \ell \leq k_2} (S_\ell \oplus S_\ell^*)^{e_\ell} \quad \bigoplus_{k_2 < \ell \leq k_3} (S_\ell \oplus S_\ell^*)^{e_\ell}$$

Where the $(S_\ell)_{1 \leq \ell \leq k_1}$ are orthogonal, the $(S_\ell \oplus S_\ell^*)_{k_1 < \ell \leq k_2}$ symplectic and the $(S_\ell \oplus S_\ell^*)_{k_2 < \ell \leq k_3}$ are general.

To end this section we will show that there is a close connection between orthogonal, symplectic and general representations

Theorem 4.6.

- (1) If $V \cong W \oplus W^*$ is a symplectic ϵ -simple representation of \mathbb{Q} then we can give W the structure of an ϵ^- -mixed representation of \mathbb{Q}^- that is **also orthogonal ϵ^- -simple** Here \mathbb{Q}^- is the supermixed quiver with the same underlying quiver and involutions as \mathbb{Q} but with $\epsilon^- = -\epsilon$ and $\sigma^- = -\sigma$.
- (2) If $V \cong W \oplus W^*$ is a general ϵ -simple representation of \mathbb{Q} with $\alpha_W = \alpha_{W^*}$ then we can give W the structure of an **almost ϵ^+ -mixed** simple representation of \mathbb{Q}^+ that is **not isomorphic to an ϵ^+ -mixed**. Here \mathbb{Q}^+

is the supermixed quiver with the same underlying quiver and involutions as \mathbb{Q} but with $\epsilon^+ : v \mapsto 1$ and $\sigma^+ : a \mapsto \sigma_a \epsilon_{t(a)}$.

Proof. (1) Because $W \cong W^*$, every element in V can then be written as a couple (v, v') with $v, v' \in W$. We know that $W^\perp = W$ and $W^{*\perp} = W^*$ therefore there are two bilinear forms on W such that

$$\langle (v, v'), (w, w') \rangle_V = \langle v, w' \rangle_1 + \langle v', w \rangle_2$$

The ϵ -commutativity of \langle, \rangle_V implies that

$$\langle v', w \rangle_2 = \langle w, \epsilon v' \rangle_1$$

We denote the adjoints according to \langle, \rangle_1 by \natural and \flat :

$$\forall v, w \in W : \langle v, \varphi w \rangle_1 = \langle \varphi^\flat v, w \rangle_1 \text{ and } \langle \varphi v, w \rangle_1 = \langle v, \varphi^\natural w \rangle_1.$$

Note that the ϵ -commutativity of \langle, \rangle also implies that $\epsilon^\flat = \epsilon^\natural = \epsilon^{-1}$.

If $a \in \mathbb{C}Q$ then the right adjoint of $\rho_V(a) = \rho_W(a) \oplus \rho_{W^*}(a)$ according to \langle, \rangle_V is

$$(\epsilon \rho_W(a) \epsilon)^\flat \oplus \rho_{W^*}(a)^\natural = \rho_V(a^*) = \rho_W(a^*) \oplus \rho_{W^*}(a^*)$$

so we know that $\epsilon \rho_W(a)^\flat \epsilon = \rho_W(a)^\natural = \rho_W(a^*)$.

Taking twice the V -adjoint shows us that $\rho_W(a)^{\flat\flat} = \epsilon \rho_W(a) \epsilon$.

It is not necessarily so that \langle, \rangle_1 is ϵ -commuting, so define ϕ by the equation $\langle v, w \rangle = \langle \phi w, v \rangle$. This map has the property that $\phi^\flat \phi = 1$. Now taking twice the adjoint here tell us that $\phi \rho_W(a)^{\flat\flat} \phi^{-1} = \rho_W(a) = \epsilon \rho_W(a)^{\flat\flat} \epsilon^{-1}$. This holds for all $a \in \mathbb{C}Q$ and as ρ_W is simple we can conclude that ϕ must be $\pm \epsilon$ so \langle, \rangle_1 is either ϵ -commuting or $-\epsilon$ -commuting.

The former is impossible because V is ϵ -simple and the diagonal submodule $\nabla = \{(v, v) | v \in W\}$ is nondegenerate:

$$\langle (v, v), (w, w) \rangle_V = \langle v, w \rangle_1 + \langle w, \epsilon v \rangle_1 = \langle v, w \rangle + \langle \epsilon w, \epsilon v \rangle_1 = 2 \langle v, w \rangle.$$

We can conclude that W admits a $-\epsilon$ -structure. But if we want to consider W as an $-\epsilon$ -mixed representation we also have to change σ to $-\sigma$ because otherwise the $*$ -operation on $\mathbb{C}Q$ is altered. We can conclude that W is a $-\epsilon$ representation of the supermixed quiver $(Q, \phi, -\epsilon, -\sigma)$.

- (2) Chose a bases (b_{vi}) for every vW and construct the dual bases (b_{vi}^*) in vW^* such that $\langle b_{vi}, b_{wj}^* \rangle_V = \delta_{v\phi(w)} \delta_{ij}$. We also define a bilinear form on W by $\langle b_{vi}, b_{wj} \rangle_W = \delta_{v\phi(w)} \delta_{ij}$. This form is only nondegenerate if $\alpha_W = \alpha_{W^*}$. It is also ϵ^+ -commuting and it trivially true that W is an almost ϵ^+ -mixed module. If \natural is the adjoint according to \langle, \rangle_W we can write the adjoint of V as

$$\rho_V(a^*) = (\epsilon \rho_{W^*}(a) \epsilon^{-1})^\natural \oplus \rho_W(a)^\natural.$$

So we can see that $\rho_W(a^*) = (\epsilon \rho_{W^*}(a) \epsilon^{-1})^\natural$. If W were isomorphic to an ϵ^+ -mixed module then $\rho_W(a^*) \cong \rho_W(a^*)^\natural$ but this is impossible because $\rho_W(a^*) \cong \rho_{W^*}(a^*)^\natural$ and ρ_W is not isomorphic to ρ_{W^*} as V is general.

□

5. LOCAL MIXED QUIVERS

Another tool we want to adapt to the ϵ -mixed quiver case is the Luna slice theorem and the construction of local quivers.

First of all let us recall the Luna slice theorem [5]. We will restrict to its use for group actions on a vector space. Let V be a vector space with a linear action from

an algebraic group G . If $v \in V$ has an orbit Gv which is closed in V , we can approximate the quotient $V//G$ in an étale neighborhood of v as follows. Construct the normal space which is the quotient of V by the tangent space to the orbit

$$N_v = V/T_v Gv.$$

On this space there is an action of G_v (the stabilizer of v in G) because it acts on both V and $T_v Gv$.

Theorem 5.1 (Luna Slice). There exists an étale neighborhoods U_v of $v \in V$ and U_0 of $0 \in N_v$ such that we have the following commutative diagram

$$\begin{array}{ccccc} U_v & \longrightarrow & U_v//G & \xrightarrow{et} & V//G \\ \downarrow \cong & & \downarrow \cong & & \\ U_0 \times_{G_v} G & \longrightarrow & U_0//G_v & \xrightarrow{et} & N_v//G_v. \end{array}$$

Which means that the quotient of V by G around v is locally isomorphic to the quotient of N_v by G_v around the zero.

Now suppose W is a semisimple representation of Q in $\text{Rep}(Q, \alpha)$. We can write down the decomposition of W as a direct sum of simples:

$$W \cong S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$$

The stabilizer of W in GL_α is isomorphic to $\text{GL}_{e_1} \times \dots \times \text{GL}_{e_k}$. Putting these things together we get the local quiver theorem:

Theorem 5.2 (Le Bruyn-Procesi). For a point $p \in \text{iss}(Q, \alpha)$ corresponding to a semisimple representation $W = S_1^{\oplus e_1} \oplus \dots \oplus S_k^{\oplus e_k}$, there is a quiver setting (Q_p, α_p) called the *local quiver setting* such that we have an isomorphism between an étale open neighborhood of the zero representation in $\text{iss}_{\alpha_p} Q_p$ and an étale open neighborhood of p .

Q_p has k vertices corresponding to the set $\{S_i\}$ of simple factors of W and the number of arrows from S_j to S_i equals

$$\sum_{a \in Q_1} \alpha_{h(a)}^i \alpha_{t(a)}^j - \sum_{v \in Q_0} \alpha_v^i \alpha_v^j + \delta_{ij}$$

where α^i is the dimension vector of the simple component S_i .

The dimension vector α_p maps every simple S_i to its multiplicity e_i .

We can extend this theorem to supermixed settings. First we note that if $W = W^*$ then the stabilizer of W is closed under the involution.

$$g \in \text{Stab}_W \Rightarrow gWg^{-1} = W \Rightarrow g^{-1*}Wg^* = W \Rightarrow g^{-1*} \in \text{Stab}_W \Rightarrow g^* \in \text{Stab}_W.$$

The same holds for the tangent space to the orbit $T_W \text{GL}_\gamma W$

$$\begin{aligned} (T_W \text{GL}_\gamma W)^* &= \{sW - Ws | s \in S_\gamma\}^* \\ &= \{-s^*W^* + W^*s^* | s \in S_\gamma\} \\ &= \{-s^*W + Ws^* | -s^* \in S_\gamma\} = T_W \text{GL}_\gamma W. \end{aligned}$$

And hence we can transport the involution on $\text{Rep}(Q, \gamma)$ to the normal space $\text{Rep}(Q, \gamma)/T_W \text{GL}_\gamma W$.

Theorem 5.3. For a point $p \in \text{Diss}(\mathbb{Q}, \gamma)$ corresponding to a semisimple representation

$$W = \bigoplus_{1 \leq \ell \leq k_1} S_\ell^{e_\ell} \bigoplus_{k_1 < \ell \leq k_2} (S_\ell \oplus S_\ell^*)^{e_\ell} \bigoplus_{k_2 < \ell \leq k_3} (S_\ell \oplus S_\ell^*)^{e_\ell}$$

there is a supermixed quiver setting (\mathbb{Q}_p, γ_p) called the *local mixed quiver setting* such that we have an isomorphism between an étale open neighborhood of the zero representation in $\text{Diss}(\mathbb{Q}_p, \gamma_p)$ and an étale open neighborhood of p .

Q_p has $2k_3 - k_2$ vertices corresponding to the set $Q_{p0} = \{S_i, S_i^*\}/\cong$ of isomorphism classes of simple factors of W . The new mixing factor gives the vertices in Q_{p0} the same type as the corresponding ϵ -mixed representations:

$$\begin{aligned} \epsilon_p(S_i) = \epsilon(S_i^*) &= \begin{cases} -1 & i \in [k_1 + 1, k_2] \\ +1 & i \notin [1, k_1] \cup [k_2 + 1, k_3] \end{cases} \\ \gamma_p(S_i) = \gamma_p(S_i^*) &= \begin{cases} \Lambda_{2e_\ell} & i \in [k_1 + 1, k_2] \\ \text{id}_{e_\ell} & i \notin [1, k_1] \cup [k_2 + 1, k_3] \end{cases} \end{aligned}$$

The number of arrows from Y to $X \in Q_{p0}$ equals

$$A_{XY} := \sum_{a \in Q_1} \alpha_{h(a)}^X \alpha_{t(a)}^Y - \sum_{v \in Q_0} \alpha_v^X \alpha_v^Y + \delta_{XY}$$

If $X = Y^*$ the number of symmetric (antisymmetric arrows) equals

$$\frac{A_{XY} \pm \left(\sum_{a=\phi(a)} \sigma_a \alpha_{h(a)}^X + \sum_{v=\phi(v)} \epsilon_v \alpha_v^X - \epsilon_X \delta_{XY} \right)}{2}$$

The $+$ -sign is used for symmetric arrows and the $-$ -sign for antisymmetric arrows. In these expressions α^X is the dimension vector associated to the representation X .

Proof. Suppose $W = W^*$ is a semisimple representation of (\mathbb{Q}, γ) with decomposition

$$\begin{aligned} W &= \bigoplus_{1 \leq \ell \leq k_1} S_\ell^{e_\ell} \bigoplus_{k_1 < \ell \leq k_2} (S_\ell \oplus S_\ell^*)^{e_\ell} \bigoplus_{k_2 < \ell \leq k_3} (S_\ell \oplus S_\ell^*)^{e_\ell} \\ &= \bigoplus_{1 \leq \ell \leq k_1} \mathbb{C}^{e_\ell} \otimes S_\ell \bigoplus_{k_1 < \ell \leq k_2} \mathbb{C}^{e_\ell} \otimes (S_\ell \oplus S_\ell^*) \bigoplus_{k_2 < \ell \leq k_3} \mathbb{C}^{e_\ell} \otimes (S_\ell \oplus S_\ell^*). \end{aligned}$$

Every S_ℓ has a dimension vector which we will denote by α^ℓ . From the previous section we know that S_ℓ itself is ϵ -commuting for $\ell \leq k_1$ so we can also find a γ^ℓ with $\bar{\gamma}^\ell = \alpha^\ell$. The same can be done for S_ℓ if $k_1 < \ell \leq k_2$, only we have to keep in mind that these γ^ℓ will be ϵ^- -commuting. Finally we set $\gamma^\ell : v \rightarrow \text{id}_{\alpha_v^\ell}$ and $\gamma^{\ell*} : v \rightarrow \epsilon_v \text{id}_{\alpha_v^\ell}$ if $\ell > k_2$. These will not give us a supermixed setting because it might be that $\alpha_v \neq \alpha_{\phi(v)}$.

These γ^ℓ can be used to construct the γ for W :

$$\gamma = \bigoplus_{1 \leq \ell \leq k_1} \text{id}_{e_\ell} \otimes \gamma^\ell \bigoplus_{k_1 < \ell \leq k_2} \begin{pmatrix} 0 & -\text{id}_{e_\ell} \\ \text{id}_{e_\ell} & 0 \end{pmatrix} \otimes \gamma^\ell \bigoplus_{k_2 < \ell \leq k_3} \text{id}_\ell \otimes \begin{pmatrix} 0 & \gamma^\ell \\ \gamma^{\ell*} & 0 \end{pmatrix}$$

If we introduce ϵ_p and γ_p as they are defined in the theorem, we can rewrite

$$\gamma = \bigoplus_{1 \leq \ell \leq k_2} \gamma_p(S_\ell) \otimes \gamma^\ell \bigoplus_{k_2 < \ell \leq k_3} \begin{pmatrix} 0 & \gamma_p(S_\ell) \otimes \gamma^\ell \\ \gamma_p(S_\ell^*) \otimes \gamma^{\ell*} & 0 \end{pmatrix}.$$

The basis for which this expression holds has the form $b_{\ell\mu} \otimes c_{\ell\nu}$ where $b_{\ell\mu}$ is the basis for \mathbb{C}^{e_ℓ} while $c_{\ell\nu}$ forms the basis for S_ℓ ($\ell \leq k_1$) or $(S_\ell \oplus S_\ell^*)$. We indexed the latter by two indices: the vertices in \mathbb{Q} and for every $v \in [1, \dots, \bar{\gamma}_v^\ell + \gamma_v^{\ell*}]$.

To every basis element corresponds a projection operator which we will denote by $\beta_{\ell\mu} \otimes \varsigma_{\ell\nu\nu} \in S_\gamma$.

Let $C(W) = \{a \in S_\gamma | aW = Wa\}$ be the centralizer of W . By Schur's lemma and the fact that $\mathbb{C}Q$ only acts on the right hand part of the basis $b_{\ell\mu} \otimes c_{\ell\nu}$, we can conclude that $C(W)$ looks like

$$\prod_{1 \leq \ell \leq k_1} \text{Mat}_{e_\ell} \otimes \text{id}_{S_\ell} \prod_{k_1 < \ell \leq k_2} \text{Mat}_{2e_\ell} \otimes \text{id}_{S_\ell} \prod_{k_2 < \ell \leq k_3} \text{Mat}_{e_\ell} \otimes \text{id}_{S_\ell} \times \text{Mat}_{e_\ell} \otimes \text{id}_{S_\ell^*}$$

which is isomorphic with S_{γ_p} . The involution is also isomorphic to the one on S_{γ_p} :

$$\begin{aligned} s^* &= \left(\bigoplus A_\ell \otimes \text{id}_{S_\ell} \bigoplus_\ell B_\ell \otimes \text{id}_{S_\ell} \bigoplus C_\ell \otimes \text{id}_{S_\ell^*} \right)^* \\ &= \bigoplus \gamma_{p\ell} A_\ell^\top \gamma_{p\ell}^{-1} \otimes \text{id}_{e_\ell} \bigoplus_\ell C_\ell^\top \otimes \text{id}_{S_\ell^*} \bigoplus B_\ell^\top \otimes \text{id}_{S_\ell}. \end{aligned}$$

.

To calculate the arrows in the local quiver we first need a lemma that deals with restriction of dualizing structures

Lemma 5.4. Let (S, M) be any dualizing structure such that S is isomorphic to the S_γ from above and denote the mixed quiver to which M corresponds Q_M . The structure of the restricted dualizing structure (S_{γ_p}, M) corresponds to a new quiver Q'_M with $2k_3 - k_2$ vertices and every arrow a in Q_M transforms for every pair of vertices $X, Y \in Q_{p0}$ into $\alpha_h^X(a) \alpha_t^Y(a)$ arrows from Y and X . If a is an (anti)-symmetric arrow and $X = Y^*$ then

$$\frac{\alpha_{h(a)}^X{}^2 + \alpha_{h(a)}^X}{2}$$

are of the same type as a while the rest of the arrows $(\frac{\alpha_{h(a)}^X{}^2 - \alpha_{h(a)}^X}{2})$ are of the opposite type of a .

Proof. We are interested in the structure of the restricted dualizing structure (S_{α_p}, M) . Let a be an arrow in M and denote its corresponding simple sub- S_α -bimodule by M_a . As an S_{γ_p} bimodule M_a decompose as a direct sum of S_{γ_p} -bimodules

$$\begin{aligned} M_a &= 1M_a1^* (1 = \bigoplus \beta_{\ell\mu} \otimes \varsigma_{\ell\nu\nu} = \bigoplus \text{id}_{e_\ell} \otimes \varsigma_{\ell\nu\nu}) \\ &= \bigoplus_{\ell_1, \ell_2} \bigoplus_{\mu_1, \mu_2} (\text{id}_{e_{\ell_1}} \otimes \varsigma_{\ell_1 h(a) \mu_1}) M_a (\text{id}_{e_{\ell_2}} \otimes \varsigma_{\ell_2 t(a) \mu_2})^* \end{aligned}$$

Note that we only need the ς 's for which the vertex $v = h(a), t(a)$ because the others act as zero on M_a . All these components are simple S_{γ_p} -bimodules and hence represent arrows from the vertex $Y = S_{\ell_2}$ to $X = S_{\ell_1}$. So in total there are

$$\sum_a \alpha_{h(a)}^X \alpha_{t(a)}^Y$$

between Y and X in Q' .

Under the involution the only components which are mapped onto themselves are the ones for which $a = \pm a^*, \ell_1 = \ell_2 (= \ell)$ and $\mu_1 = \mu_2 (= \mu)$.

The action under the involution on $x \in (\text{id}_{e_\ell} \otimes \varsigma_{\ell h(a)\mu}) M_a (\text{id}_{e_\ell} \otimes \varsigma_{\ell h(a)\mu})^*$ is given by

$$\begin{aligned}
x &\mapsto \sigma_a \epsilon_{t(a)} \gamma \left((\text{id}_{e_\ell} \otimes \varsigma_{\ell h(a)\mu}) x (\text{id}_{e_\ell} \otimes \varsigma_{\ell h(a)\mu})^* \right)^{-1} g \\
&= \sigma_a \epsilon_{t(a)} \underbrace{(\text{id}_{e_\ell} \otimes \varsigma_{\ell h(a)\mu}) \gamma (\text{id}_{e_\ell} \otimes \varsigma_{\ell h(a)\mu})^*}_{\epsilon_{h(a)} \epsilon_{S_\ell} (\gamma_\ell \otimes \varsigma_{\ell h(a)\mu})} x^\top \underbrace{(\text{id}_{e_\ell} \otimes \varsigma_{\ell h(a)\mu})^\top \gamma^{-1} (\text{id}_{e_\ell} \otimes \varsigma_{\ell h(a)\mu})^*}_{(\gamma_\ell^{-1} \otimes \varsigma_{\ell h(a)\mu}^*)} \\
&= \sigma_a \underbrace{\epsilon_{h(a)} \epsilon_{t(a)}}_{=1} \epsilon_{S_\ell} (\gamma_\ell \otimes \text{id}) x (\gamma_\ell^{-1} \otimes \text{id})
\end{aligned}$$

We can conclude that $(\alpha_{h(a)}^X)^2 - \alpha_{h(a)}^X$ of the arrows in which $a = \phi(a)$ decomposes are not mapped to themselves under the involution $M_b \neq M_b^*$. Taking linear combinations $b - b^*$ and $b + b^*$ we can make this equivalent to $\frac{(\alpha_{h(a)}^X)^2 - \alpha_{h(a)}^X}{2}$ symmetric and an equal number antisymmetric arrows.

The calculation above adds to these equal amounts $\alpha_{h(a)}^X$ more symmetric or antisymmetric depending on the type of a . \square

As S_{γ_p} -bimodules with dualizing structure we have that $N_p = \text{Rep}(\mathbb{Q}, \gamma)/T_W \text{GL}_\gamma W$. Also we can identify $T_W \text{GL}_\gamma W$ with S_γ/S_{γ_p} but this identification only works as bimodules without the dualizing structure. In order to make it work with the dualizing structure we must put a new dualizing structure on S_γ as an S_γ -bimodule. This is done by putting $\star : S_\gamma \rightarrow S_\gamma : x \rightarrow -x^*$. This new structure turns the morphism

$$\pi : S_\gamma \rightarrow T_W \text{GL}_\gamma W : x \mapsto xW - Wx$$

into a \star -morphism: $\pi(x^*) = x^*W - Wx^* = -x^*W + Wx^* = -x^*W^* + W^*x^* = \pi(x)^*$. In quiver terminology Q_{S_γ} is a quiver with the same number of vertices as Q but with a unique loop in every vertex. The loops in orthogonal vertices are antisymmetric, while those in symplectic vertices are symmetric.

So to determine the total number of arrows in N_p from Y to X we can use the following formula

$$\begin{aligned}
A_{XY} &:= \#\{X \leftarrow Y \text{ in } Q'_{N_p}\} \\
&= \#\{X \leftarrow Y \text{ in } Q'_{\text{Rep}(\mathbb{Q}, \gamma)}\} - \#\{X \leftarrow Y \text{ in } Q'_{S_\gamma}\} + \#\{X \leftarrow Y \text{ in } Q'_{S_{\gamma_p}}\} \\
&= \sum_{a \in Q_1} \alpha_{h(a)}^X \alpha_{t(a)}^Y - \sum_{v \in Q_0} \alpha_v^X \alpha_v^Y + \delta_{XY}
\end{aligned}$$

To determine the number of symmetric arrows from Y to X we can do the same thing:

$$\begin{aligned}
\#\{X \overset{\leftarrow}{\vdash} Y \text{ in } Q'_{N_p}\} &= \#\{X \overset{\leftarrow}{\vdash} Y \text{ in } Q'_{\text{Rep}(\mathbb{Q}, \gamma)}\} - \#\{X \overset{\leftarrow}{\vdash} Y \text{ in } Q'_{S_\gamma}\} + \#\{X \overset{\leftarrow}{\vdash} Y \text{ in } Q'_{S_{\gamma_p}}\} \\
&= \frac{A_{XY} + \sum_{a=\phi(a)} \sigma_a \alpha_{h(a)}^X - \sum_{v=\phi(v)} (-\epsilon_v) \alpha_v^X + (-\epsilon_X) \delta_{XY}}{2}
\end{aligned}$$

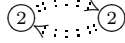
for the antisymmetric arrows we obtain

$$\frac{A_{XY} - \sum_{a=\phi(a)} \sigma_a \alpha_{h(a)}^X + \sum_{v=\phi(v)} (-\epsilon_v) \alpha_v^X - (-\epsilon_X) \delta_{XY}}{2}$$

\square

We will illustrate the theorem with a few examples.

- Consider the setting



where the two vertices are each others dual and all arrows are antisymmetric. If we consider a representation for which every arrow is represented by Λ_2 . This representation is the direct sum of two isomorphic representation with dimension vector 1. These two are each others dual so the local quiver setting is



with 3 loops of which there are $\frac{3+(-4+1)}{2} = 0$ symmetric arrows and 3

- The setting



where the upper vertex is orthogonal and the lower vertex is symplectic. Now we look at the local quiver setting of the representation that assigns to the two left arrows the identity matrices (and to the right arrows $-\Lambda_2$).

The stabilizer of this setting is $GL_1 \times GL_1$ and the local quiver setting looks like



As one can calculate the number of symmetric arrows from the left vertex to the right is $\frac{2+0-0+0}{2} = 1$

6. SIMPLES

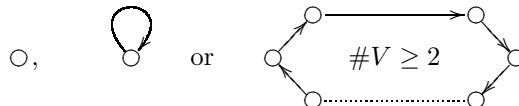
In this section we are going to describe a method to determine whether a given supermixed quiver setting (Q, γ) has a dualizing space, $\text{DRep}(Q, \gamma)$, that contains simples and if it does what kind of simples.

To answer this question we first need to recall when an ordinary quiver setting whose representation space simple representations. This question was solved by Le Bruyn and Procesi in [3]. To state this result we need some terminology.

A quiver setting (Q, α) is *sincere* if all vertices have a nonzero dimension (i.e. $\forall v \in Q_0 : \alpha_v > 0$). A quiver Q is *strongly connected* if for every pair of vertices $v, w \in Q_0$ there are paths $v \leftarrow w$ and $w \leftarrow v$. The *support* of a setting is the subquiver that contains only the vertices with $\alpha_v > 0$. The support of a representation is the same as the support of its setting.

Theorem 6.1 (Le Bruyn, Procesi). Let (Q, α) be a sincere quiver setting. There exist simple representations of dimension vector α if and only if

- If Q is of the form



and $\alpha = 1$ (this is the constant map from the vertices to 1).

- Q is not of the form above, but strongly connected and

$$\forall v \in V : \alpha_v \leq \sum_{h(a)=v} \alpha_{t(a)} \text{ and } \alpha_v \leq \sum_{t(a)=v} \alpha_{h(a)}$$

If (Q, α) is not sincere, the simple representations classes are in bijective correspondence to the simple representations classes of its support.

In all cases except for the one vertex without loops there are an infinite number of isomorphism classes of simples with that dimension vector. In the case of the one vertex v without loops, there is one unique simple representation T_v .

As stated above we have a nice numerical criterion to check whether a given setting has simples or not. The question for supermixed quiver settings is a bit more complicated because we have 3 types of simples. However it suffices to check for every (Q, γ) there exist orthogonal simples and if so whether all simples in $\text{Rep}(Q, \gamma)$ are isomorphic to an orthogonal ϵ -simple.

Theorem 6.2. The following pairs of statements are equivalent:

- $\text{DRep}(Q, \Lambda_2 \otimes \gamma)$ contains symplectic ϵ -simples.
- $\text{DRep}(Q^-, \gamma)$ contains orthogonal ϵ -simples.

and

- $\text{DRep}(Q, \gamma)$ contains general ϵ -simples.
- One of the two possibilities below holds:
 - (1) $\gamma \cong \text{id}_2 \otimes \gamma'$ and $\text{Rep}(Q^+, \gamma')$ contains simples that are not isomorphic to an orthogonal ϵ -simple.
 - (2) There exists a dimension vector α with $\alpha + \alpha^* = \bar{\gamma}$ such that $\text{Rep}(Q, \alpha)$ contains simples and $\alpha \neq \alpha^*$.

Proof. These statements follow from theorem 4.6. □

Given the solution to the first question, (are there orthogonal simples), the answer to the second questions is quite straitforward:

Theorem 6.3. If $\text{DRep}(Q, \gamma)$ contains orthogonal simples then every simple in $\text{Rep}(Q, \gamma)$ is isomorphic to an orthogonal simple if and only if

$$\dim \text{iss}(Q, \gamma) = \dim \text{Diss}(Q, \gamma).$$

This condition can be expressed numerically as

$$\sum_{a \in Q_1} \alpha_{h(a)} \alpha_{t(a)} - \sum_{v \in Q_0} \alpha_v^2 + 1 = \frac{1}{2} \left(\sum_{a \in Q_1} \alpha_{h(a)} \alpha_{t(a)} - \sum_{v \in Q_0} \alpha_v^2 + \sum_{\phi(a)=a} \sigma_a \alpha_{h(a)} - \sum_{\phi(v)=v} \epsilon_v \alpha_v \right).$$

with $\alpha = \bar{\gamma}$.

Proof. Because $\text{Diss}(Q, \gamma)$ is a closed subset of $\text{iss}(Q, \gamma)$, the dimension condition implies that both spaces are identical. This means that every representation in $\text{Rep}(Q, \gamma)$ is isomorphic to a representation in $\text{DRep}(Q, \gamma)$. A simple representation that is identical to its dual is an orthogonal simple.

Now vice versa suppose every simple in $\text{iss}(\mathbb{Q}, \gamma)$ is isomorphic to an orthogonal then we know that as the simples form a open subset of $\text{iss}(\mathbb{Q}, \gamma)$, the dimension of $\text{Diss}(\mathbb{Q}, \gamma)$ must be at least the dimension of $\text{iss}(\mathbb{Q}, \gamma)$.

The numerical condition comes from the calculations

$$\begin{aligned} \dim \text{iss}(\mathbb{Q}, \alpha) &= \dim \text{Rep}(\mathbb{Q}, \alpha) - \dim GL_\alpha + \dim \text{stab}(\text{simple}) \\ &= \sum_{a \in Q_1} \alpha_{h(a)} \alpha_{t(a)} - \sum_{v \in Q_0} \alpha_v^2 + 1 \\ \dim \text{Diss}(\mathbb{Q}, \gamma) &= \dim \text{DRep}(\mathbb{Q}, \gamma) - \dim \text{DGL}_\gamma + \dim \text{stab}(\text{orth. simple}) \\ &= \frac{1}{2} \left(\sum_{a \in Q_1} \alpha_{h(a)} \alpha_{t(a)} + \sum_{\phi(a)=a} \sigma_a \alpha_{h(a)} - \sum_{v \in Q_0} \alpha_v^2 - \sum_{\phi(v)=v} \epsilon_v \alpha_v \right). \end{aligned}$$

□

The ideal answer to the existence question for simples would be a numerical criterion like the one from [3]. This becomes a combinatorial nightmare because the three different vertex types give different conditions. Therefore we will give an algorithmic approach. To clarify what is meant by that we will first transform the ordinary theorem into an algorithmic result.

Before we proceed to prove this algorithm works we need a lemma

Lemma 6.4. If W is a semisimple representation in $\text{Rep}(\mathbb{Q}, \alpha)$ and (Q_L, α_L) is its local quiver setting then $\text{Rep}(\mathbb{Q}, \alpha)$ contains simple representations if and only if $\text{Rep}(Q_L, \alpha_L)$ contains simples

Proof. The simples in $\text{Rep}(\mathbb{Q}, \alpha)$ or in $\text{Rep}(Q_L, \alpha_L)$ form a Zariski open part and the étale isomorphism will map simples to simples because it preserves the closedness of orbits and the size of the stabilizers. □

If W is not a representation with dimension vector α but $\alpha^W \leq \alpha$ then we can still consider a local quiver setting of *the completed representation* \hat{W} which is obtained by adding the standard simples

$$\hat{W} := W \oplus \bigoplus_{v \in Q_0} T_v^{\oplus \alpha_v - \alpha_v^W}.$$

Theorem 6.5 (reformulation of theorem 6.1). To check whether $\text{Rep}(\mathbb{Q}, \alpha)$ contains simples follow the following algorithm.

- S0 If (\mathbb{Q}, α) is not sincere then $\text{Rep}(\mathbb{Q}, \alpha)$ contains simples if and only if $\text{Rep}(\mathbb{Q}', \alpha')$ does. Here (\mathbb{Q}', α') is support of (\mathbb{Q}, α) .
- S1 If $\forall v \in Q_0 : \alpha_v \neq 0$ and \mathbb{Q} is not strongly connected then $\text{Rep}(\mathbb{Q}, \alpha)$ does not contain simples. If \mathbb{Q} has one vertex and no arrows then $\text{Rep}(\mathbb{Q}, \alpha)$ contains no simples if $\alpha > 1$.
- S2 If $\alpha = 1$ then $\text{Rep}(\mathbb{Q}, \alpha)$ contains simples if and only if \mathbb{Q} is strongly connected.
- S3 If $\alpha \neq 1$ and \mathbb{Q} is strongly connected, then by S2, there exists a simple W with dimension vector 1. $\text{Rep} \mathbb{Q}, \alpha$ contains simples if and only if the local quiver setting (Q_L, α_L) of the completed representation \hat{W} contains simples.

Proof. The step S0 is trivial. Step S1 follows from the fact that every strongly connected component gives rise to a subrepresentation. If Q is strongly connected every $\alpha = \text{id}$ -dimensional representation that assigns to every arrow a nonzero scalar is simple (S2). Step S3 is a consequence of lemma 6.4

Finally we have to show that the recursion will end in a finite number of steps. This is because every S3-step reduces the dimension of all vertices with one and adds one vertex with dimension 1. \square

Now we want to generalize the theorem above to the case of orthogonal simples. First we can generalize the lemma about the local quivers:

Lemma 6.6. If W is a semisimple ϵ -mixed representation in $\text{DRep}(\mathbb{Q}, \gamma)$ and (Q_L, γ_L) is its local quiver setting then $\text{DRep}(Q, \gamma)$ contains simple representations if and only if $\text{DRep}(\mathbb{Q}_L, \gamma_L)$ contains simples

Proof. Analogous to the ordinary case. \square

We also need some lemmas about supermixed settings with special dimension vectors.

Lemma 6.7. Suppose \mathbb{Q} is a supermixed quiver that can be written as $c \cup c^*$ where c is a cycle. Then we can find an ϵ -mixed ϵ -simple representation W such that its dimension vector is 1 or 2.

Proof. First assume that $c \cap c^*$ is empty then all arrows and vertices are of general type. Consider the representation W that assigns to every arrow the scalar $1 \in \text{Mat}_{1 \times 1}(\mathbb{C})$. This representation is the direct sum of two simples corresponding to the two cycles. The involution swaps these two simples so W is a general ϵ -simple.

Now assume that $c \cap c^*$ is non-empty but that it does not contain symplectic vertices or antisymmetric arrows. In that case we can again assign the scalar $1 \in \text{Mat}_{1 \times 1}(\mathbb{C})$ to every arrow. This representation will be simple because it is strongly connected and hence W is an orthogonal ϵ -simple.

Finally if \mathbb{Q} contains a symplectic vertex or an antisymmetric arrow, We chose a minimal set of arrows C such that C meets every $\{a, \phi(a)\}$. Let W be the representation that assigns the 2×2 -identity matrix to every arrow of C that is not selfdual. To the selfdual arrows we assign the identity matrix if $a^* = a$ and the standard symplectic if $a^* = -a$. W is semisimple because all arrows are invertible and $C \cup \phi(C)$ is strongly connected. The stabilizer DGL_W of W inside DGL_γ contains

$$\{g \in \text{DGL}_\gamma \mid \forall v, w \in \mathbb{Q}'_0 : g_v = g_w \in O_2 \cap \text{Sp}_2\} \cong \text{GL}_1$$

but because all arrows are invertible the value of g on a vertex v fixes all other values:

$$g_w = W_p g_v W_p^{-1} \text{ if } p : v \leftarrow w,$$

so $\text{DGL}_W \subset \text{DGL}_{\bar{\gamma}_v} = \text{Sp}_2$ if v is the symplectic vertex or $\text{DGL}_W \subset \{h \in \text{DGL}_{\bar{\gamma}_{h(a)}} \mid h \Lambda_2 h^T = \text{Sp}_2 \text{ if } a^* = -a\}$.

This means that there are two possibilities for DGL_W : either it is GL_1 in which case W is a general ϵ -simple or it is Sp_2 in which case W is a symplectic ϵ -simple. \square

Remark 1. Note that in all cases except for the quiver with just one (symplectic) vertex and only antisymmetric loops, the corresponding local quiver setting has less vertices of dimension 2 than the original.

Definition 6.1. Suppose (\mathbb{Q}, γ) is a strict supermixed quiver setting such that

- (1) $\forall v \in Q_0 : 1 \leq \bar{\gamma}_v \leq 2$.
- (2) $\bar{\gamma}_v = 2$ if v is a symplectic vertex or $v = h(a), t(a)$ for an antisymmetric arrow a .
- (3) No cycle has dimension bigger than 1.
- (4) $\text{Rep}(\mathbb{Q}, \gamma)$ contains simples.

Such a supermixed quiver settings are called *basic* and they will replace the quiver settings with dimension vector 1 in the ordinary case.

Lemma 6.8. If (Q, γ) is basic then $\text{DRep}(\mathbb{Q}, \gamma)$ contains orthogonal ϵ -simples if and only if for every pair of vertices (v, w) of dimension one there is a path $p : v \rightarrow w$ that is not antisymmetric $p^* \neq -p$.

Proof. The condition is necessary because if there is no such path from v to w then the $\mathbb{C}QvW$ is a subrepresentation but $wW \cap \mathbb{C}QvW = 0$ because every path from v to w is antisymmetric and hence evaluates to zero.

Now we prove that the condition is also sufficient. First let us define some open subset of $\text{DRep}(\mathbb{Q}, \gamma)$.

- For every primitive path $p (\neq -p^*)$ between two vertices of dimension one we can consider the open subset $U_p \subset \text{DRep}(\mathbb{Q}, \gamma)$ of all representations W such that W_p is invertible. These subsets are nonempty: there is a surjective S_γ -bimodule morphism

$$\begin{cases} \text{Rep}(\mathbb{Q}, \gamma) \rightarrow L_1 = \mathbb{C}^2 : W \mapsto (W_p, W_{p^*}) & p \neq p^* \\ \text{Rep}(\mathbb{Q}, \gamma) \rightarrow L_2 = \mathbb{C} : W \mapsto (W_p) & p = p^* \end{cases}$$

In both cases the dualizing space is nonzero ($D(L_1) = \{(\lambda, \lambda) | \lambda \in \mathbb{C}\}$, $D(L_2) = \mathbb{C}$), so U_p must also be nonzero.

- For every vertex v of dimension 2 we can consider the open subset $U_v \subset \text{DRep}(\mathbb{Q}, \beta)$ of all representations W such that both $\oplus_{h(a)=v} W_a$ and $\oplus_{t(a)=v} W_a$ have rank 2. It is easy to see that U_v is also non-empty.

Suppose that $W \in \cap_p U_p \cap_v U_v$ then W is a simple representation. If $W' \subset W$ is a simple subrepresentation then it cannot be concentrated in one vertex v with $\bar{\gamma}_v = 2$ because W is inside the U_v . It also cannot be concentrated in only vertices with dimension $\alpha_v = 2$ because there is no cycle of this kind of vertices. So its support contains a vertex of dimension 1 and because $W \in \cap_p U_p$ it is supported in all vertices of dimension one.

If W were not simple its semisimplification W^{ss} would be the sum of simples in W , but every simple is supported in all vertices of dimension one it is not possible that W^{ss} has more than one summand and hence W must be simple. \square

Theorem 6.9. Let (\mathbb{Q}, γ) be a sincere supermixed quiver setting. To check whether $\text{DRep}(\mathbb{Q}, \gamma)$ contains orthogonal simples follow the following algorithm.

- 00 $\text{DRep}(\mathbb{Q}, \gamma)$ contains simples if and only if $\text{DRep}(\mathbb{Q}', \alpha')$ does. Here (\mathbb{Q}', α') is the setting obtained by deleting all the antisymmetric arrows of dimension one and all the antisymmetric loops on symplectic vertices of dimension 2.

- O1 If $\text{Rep}(\mathbb{Q}, \gamma)$ does not contain simples or there exists a pair of vertices with dimension 1 such that every path $p : v \rightarrow w$ is antisymmetric then $\text{DRep}(\mathbb{Q}, \gamma)$ does not contain orthogonal ϵ -simples.
- O2 If (\mathbb{Q}, γ) is basic then we apply lemma 6.8.
- O3 If (\mathbb{Q}, γ) is not basic then either
 - A. If (Q, γ) contains a cycle c with dimension at least 2, the setting $(c \cup c^*, 1 \text{ or } 2)$ has an ϵ -simple representation of the form in lemma 6.7. Consider the local supermixed setting the completed representation \hat{V} . $\text{Rep}(\mathbb{Q}, \gamma)$ contains simples if and only if this local setting does.
 - B. If all cycles have dimension one we consider a dimension vector β such that (Q, β) satisfies the conditions of lemma 6.8, and let V be an orthogonal ϵ -simple for this setting. Then (\mathbb{Q}, γ) contains orthogonal simples and (Q, α) contains simples if and only if the local setting generated by \hat{V} does.

Proof. Step O0 works because antisymmetric arrows of dimension one are zero and symmetric loops on 2-dimensional symplectic vertices are scalar matrices. Step O1 follows the line of the first paragraph in the proof of lemma 6.8. So now we have to prove that after a finite number of O3-steps we end in a basic setting or a non-simple one. Step O3A reduces the dimensions of the higher dimensional cycles so after a finite steps there will be no higher-dimensional cycles (see remark 1 and observe that by O0 there are no antisymmetric loops on symplectic vertices with dimension 2). Every simple setting (Q, α) that has passed step O0 will contain a basic subsetting the sum of the dimensions in local subsetting will be strictly lower than the sum $\sum_{v \in Q_0} \alpha_v$, so the algorithm will end after a finite number of O3B-steps. \square

We will now give an easy application of this algorithm to the easiest quiver settings: the ones with one vertex.

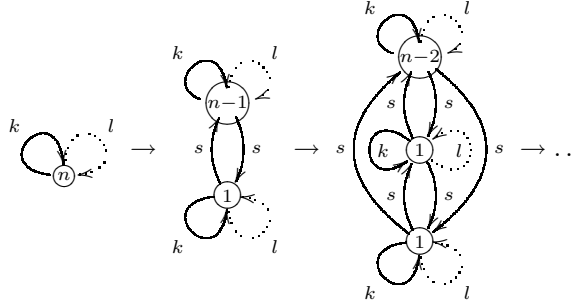
Theorem 6.10. The dualizing space of a supermixedquiver setting with one vertex, k symmetric loops and l antisymmetric loops contains orthogonal simples if and only if

- (1) the vertex v is orthogonal with dimension n and
 - (a) if $n = 1$
 - (b) if $n = 2$, $k + l \geq 2$ and $k \geq 1$,
 - (c) if $n \geq 3$ and $k + l \geq 2$.
- (2) the vertex v is symplectic with dimension $n \in 2\mathbb{N}$ and
 - (a) if $n = 2$ and $k \geq 2$
 - (b) if $n = 4, 6$, $k + l \geq 2$ and $k \geq 1$ or $l \geq 3$
 - (c) if $n \geq 8$ and $k + l \geq 2$.

Proof. First note that if $n > 1$ then $k + l \geq 2$ because otherwise the underlying quiver setting does not contain simple representations in $\text{Rep}(Q, \alpha)$ by theorem 6.1. If $n = 1$ then every representation is simple and hence every representation in $\text{DRep}(\mathbb{Q}, \gamma)$ is orthogonal ϵ -simple.

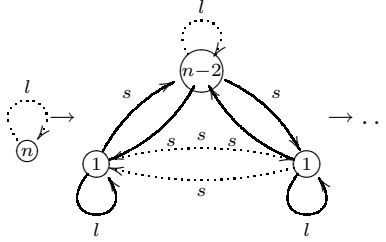
Now let us first concentrate on the orthogonal case. If we start applying the algorithm the step O3A will do the following

- If $k \geq 1$ we take c to be a symmetric loop.



here $s = k + l - 1$. So after n steps we get a quiver with only orthogonal vertices of dimension 1 and s arrows between every pair of vertices. This setting is basic and hence contains ϵ -simples.

- If $k = 0$ we must take c to be an antisymmetric loop.



So after $\frac{n}{2}$ steps we get a quiver with only orthogonal vertices of dimension 1 and $l - 1$ arrows between every pair of vertices. The arrows between a vertex and its dual are antisymmetric and can be deleted. Therefore the setting is only basic if $n > 2$.

For the symplectic case we can do the same thing but again we have to differentiate between $k = 0$ and $k \geq 1$.

- If $k \geq 1$ the representation we split off corresponding to this symmetric loop in $O3A$ is a general ϵ -simple. This is the same as for the orthogonal case with $k = 0$, but now there is at least one symmetric arrow between two dual vertices and hence the setting is always basic.
- If $k = 0$ then $l \geq 2$ and the representation we split of first in $O3A$ is a symplectic ϵ -simple. Then we can split of an ϵ -simple with corresponding to a cycle that runs through the two cycles (note that such a cycle exists because $k + l - 1 = l - 1 \geq 1$). We can continue this way until the original vertex has dimension 0. Then we end up with a quiver with one symplectic vertex of dimension 2 and $l((\frac{n}{2})^2 - 1) + 1$ loops of which $\frac{l((\frac{n}{2})^2 - 1) - (l+1)\frac{n}{2} + 1}{2}$ are symmetric. This number can be obtained by using the induction procedure or more directly by observing that the last quiver setting we obtain corresponds to the local quiver setting of a symplectic simple representation of the original setting. If the last setting contains an orthogonal representation then $\frac{l((\frac{n}{2})^2 - 1) - (l+1)\frac{n}{2} + 1}{2}$ must be bigger than 1. This holds if $l \geq 3$ or if $n \geq 8$.

□

REFERENCES

1. William Crawley-Boevey, *Normality of Marsden-Weinstein reductions for representations of quivers*, Math. Ann. **325** (2003), no. 1, 55–79. MR MR1957264 (2004c:16017)
2. Harm Derksen and Jerzy Weyman, *Generalized quivers associated to reductive groups*, Colloq. Math. **94** (2002), no. 2, 151–173. MR MR1967372 (2004f:16021)
3. Lieven Le Bruyn and Claudio Procesi, *Semisimple representations of quivers*, Trans. Amer. Math. Soc. **317** (1990), no. 2, 585–598. MR MR958897 (90e:16048)
4. A. A. Lopatin and A. N. Zubkov, *Semi-invariants of mixed representations of quivers*, Transform. Groups **12** (2007), no. 2, 341–369. MR MR2323686
5. Domingo Luna, *Slices étales*, Sur les groupes algébriques, Soc. Math. France, Paris, 1973, pp. 81–105. Bull. Soc. Math. France, Paris, Mémoire 33. MR MR0342523 (49 #7269)
6. A. N. Zubkov, *Invariants of mixed representations of quivers. I*, J. Algebra Appl. **4** (2005), no. 3, 245–285. MR MR2150664 (2006g:16032)
7. ———, *Invariants of mixed representations of quivers. II. Defining relations and applications*, J. Algebra Appl. **4** (2005), no. 3, 287–312. MR MR2150665 (2006g:16033)

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